Recall: for $2 \le T \le 2X$,

$$\psi(X) = X - \sum_{|\mathrm{Im}\,\rho| \le T} \frac{X^\rho}{\rho} + O\left(\frac{X(\log X)^2}{T}\right).$$

The explicit formula shows that the distribution of primes is closely related to the location of zeros of $\zeta(s)$. Next we see what we obtain under the most optimistic assumption about the location of zeros (RH).

<u>Theorem</u> (Error term under RH)
The Riemann hypothesis is true if and only if

$$\psi(X) = X + O(X^{\frac{1}{2}+\varepsilon}), \forall \varepsilon > 0.$$

<u>Proof</u>: "$\Rightarrow$" Applying Explicit Formula with $X = T$, we have

$$\psi(X) = X - \sum_{\rho: |\mathrm{Im}\,\rho| \le X} \frac{X^\rho}{\rho} + O((\log X)^2).$$

Now suppose all non-trivial zeros have $\mathrm{Re}(\rho) = \frac{1}{2}$.
For $n \ge 5$, there are $O(\log n)$ zeros with $|\mathrm{Im}\,\rho| \in [n, n+1]$, and each one contributes $O\left(\frac{X^{1/2}}{n}\right)$.

Therefore $\psi(X) = X + O\left(1 + X^{1/2} \sum_{5 \le n \le X} \frac{\log n}{n} + (\log X)^2\right)$.

$$= x + O(x^{1/2} (\log x)^2).$$ ✓

"$\Longleftarrow$" By partial summation (initially in $\mathrm{Re}(s) > 1$)

$$-\frac{\zeta'}{\zeta}(s) = \sum \Lambda(n) n^{-s} = s \int_1^\infty \psi(y) y^{-s-1} dy.$$

Suppose $\psi(x) = x + R(x)$, where $R(x) \ll x^{\frac{1}{2}+\varepsilon}$.

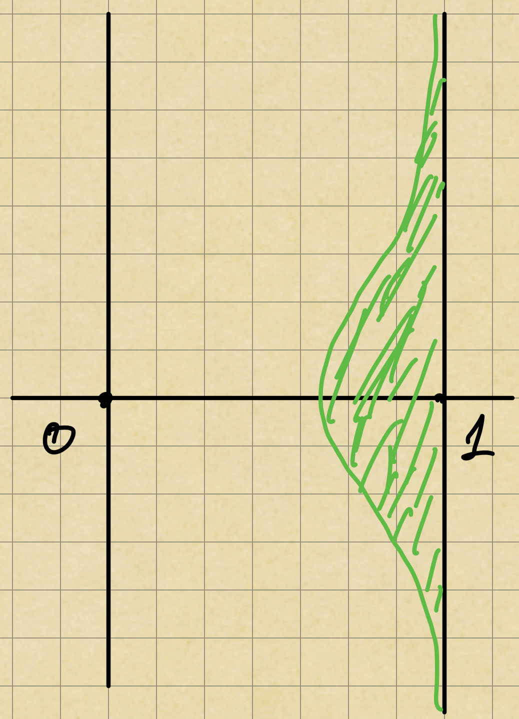
$$\Rightarrow -\frac{\zeta'}{\zeta}(s) = \frac{s}{s-1} + s \int_1^\infty R(x) x^{-s-1} dx.$$

Then integral is holomorphic for $\mathrm{Re}(s) > \frac{1}{2}+\varepsilon$, so $\zeta(s)$ is zero-free in this region. $\square$

Unfortunately RH is completely out of reach, but we have some understanding for zero-free regions.

<u>**Theorem**</u> (Zero-free region for $\zeta(s)$)

There exists a constant $c > 0$ such that if $\rho = \sigma + it$ is a non-trivial zero of $\zeta(s)$, then $\sigma \leq 1 - \frac{c}{\log(2+|t|)}$.



Proof: We have the following key elementary identity.
For $\alpha \in \mathbb{R}$, $3 + 4\cos\alpha + \cos 2\alpha = 2(1+\cos\alpha)^2 \geq 0$.

Note that for $\sigma > 1$,

$$\mathrm{Re}\left(-\frac{\zeta'}{\zeta}(\sigma+it)\right) = \mathrm{Re}\left(\sum_{n\geq 1} \frac{\Lambda(n)}{n^{\sigma+it}}\right)$$

$$= \sum_{n\geq 1} \frac{\Lambda(n)}{n^\sigma} \cos(t \log n).$$

Therefore, for $\sigma > 1$, we have

$$\mathrm{Re}\left(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma+it) - \frac{\zeta'}{\zeta}(\sigma+2it)\right)$$

$$= \sum_n \frac{\Lambda(n)}{n^\sigma}\left(\underbrace{3 + 4\cos(t\log n) + \cos(2t\log n)}_{\geq 0}\right)$$

$$\geq 0.$$

$C_0>0$ a constant we choose later

Let $1<\sigma<2$ and $t\geq C_0$. Then by partial fraction expansion of $\frac{\zeta'}{\zeta}(s)$, we have for $s=\sigma+it$

$$-\frac{\zeta'}{\zeta}(\sigma+it) = -\sum_{|s-\rho|<1}\frac{1}{s-\rho} + O(\log t).$$

Note that $\mathrm{Re}\left(\frac{1}{s-\rho}\right) = \frac{\mathrm{Re}(s-\rho)}{|s-\rho|^2} > 0$.

Also, recall that $-\frac{\zeta'}{\zeta}(\sigma) = \frac{1}{\sigma-1} + O(1)$.

Now let a zero $\rho_0 = \beta+it$, with $t\geq C_0$.

Then $0 \leq \mathrm{Re}\left(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma+it) - \frac{\zeta'}{\zeta}(\sigma+2it)\right)$

$$= \frac{3}{\sigma-1} + O(1) - \mathrm{Re}\left(\frac{4}{\sigma+it-\rho_0}\right) - \underbrace{\sum_{\substack{\rho:|\rho-s|<1\\ \rho\neq\rho_0}} \mathrm{Re}\left(\frac{1}{s-\rho}\right)}_{\geq 0}$$

$$- \underbrace{\sum_{\rho:|\rho-(\sigma+2it)|<1} \mathrm{Re}\left(\frac{1}{\sigma+2it-\rho}\right)}_{\geq 0} + O(\log t).$$

Hence there exists an absolute constant C_1 such that

$$\frac{3}{\sigma-1} - \frac{4}{\sigma-\beta} + C_1\log t \geq 0.$$

Choose $\sigma = 1 + \frac{\delta}{\log t}$, for some $\delta > 0$.

Then $\beta < 1 + \frac{\delta}{\log t} - \frac{4\delta}{(3 + C_2\delta)\log t}$

Let $\delta = \frac{1}{3C_1}$, we have $\beta < 1 - \frac{1}{15 C_1 \log t}$.

This proves theorem if $\operatorname{Im} \rho \geq C_0$. By symmetry, it follows for all ρ with $\operatorname{Im} \rho \leq -C_0$.

We have finitely many zeros with $|\operatorname{Im} \rho| \leq C_0$, and all away bounded away from $\{\operatorname{Re}(s) = 1, \operatorname{Im}(s) \leq C_0\}$.

So theorem follows if $c > 0$ sufficiently small. $\square$

(We choose C_0 such that $\zeta(1+it) \neq 0$, for $0 < |t| \leq C_0$.

Such C_0 exists because $\zeta(s)$ has pole at 1,

$$\zeta(s) = \frac{1}{s-1} + O(1).$$
) $\square$

This zero-free region is enough to prove PNT:

<u>**Theorem**</u>: There exists a constant $c_2 > 0$

$$\psi(x) = x + O\left(x e^{-c_2\sqrt{\log x}}\right).$$

Proof: We know that for $2 \le T \le X$, we have
$$\psi(X) = X - \sum_{|\operatorname{Im}\rho| \le T} \frac{X^\rho}{\rho} + O\left(\frac{X(\log X)^2}{T}\right).$$

For each ρ in the sum, we have $|X^\rho| = X^{\operatorname{Re}(\rho)} \le X^{1-\frac{c}{\log T}}$

Therefore $\psi(X) = X + O\left(X^{1-\frac{c}{\log T}} \sum_{|\operatorname{Im}\rho|\le T} \frac{1}{|\rho|}\right) + O\left(\frac{X(\log X)^2}{T}\right).$

There are $O(\log(1+n))$ zeros with $|\operatorname{Im}\rho| \in [n, n+1]$, therefore the sum is of size $O((\log T)^2)$.

Choose $T = \exp(\sqrt{\log X})$ to balance size of error terms. Then we find indeed that, for a suitable constant $c_1 > 0$
$$\psi(X) = X + O(X \exp(-c_1\sqrt{\log X})). \qquad \square$$

(say $c_1 = \min(\frac{c}{2}, \frac{1}{2})$).

Remark: Recall that for any $\varepsilon > 0$, $c > 0$, $N \in \mathbb{N}$,
$$(\log X)^N \ll \exp(c\sqrt{\log X}) \ll X^\varepsilon.$$

Corollary: (Prime number theorem)
$$\pi(X) = Li(X) + O(X \exp(-c_2\sqrt{\log X})).$$

Proof: Recall $\vartheta(x) = \sum_{p \leq x} \log p$.

We showed $\vartheta(x) = \psi(x) + O(\sqrt{x}(\log x)^2)$

$\implies \vartheta(x) = x + O(x \exp(-c_1\sqrt{\log x}))$.

By partial summation: $\pi(x) = \frac{\vartheta(x)}{\log x} + \int_{3/2}^{x} \frac{\vartheta(t)}{t(\log t)^2}\, dt$

Hence $\pi(x) = \frac{x}{\log x} + \int_{3/2}^{x} \frac{1}{(\log t)^2}\, dt + O\left(\frac{x e^{-c_1\sqrt{\log x}}}{\log x} + \int_{3/2}^{x} \frac{e^{-c_1\sqrt{\log t}}}{(\log t)^2}\, dt\right)$

Main term: Note that $\left(\frac{1}{\log t}\right)' = -\frac{1}{t(\log t)^2}$.

Therefore $\frac{x}{\log x} + \int_{3/2}^{x} \frac{1}{(\log t)^2}\, dt =$

$= \frac{x}{\log x} + \left[\frac{-t}{\log t}\right]_{3/2}^{x} + \int_{3/2}^{x} \frac{1}{\log t}\, dt = \mathrm{Li}(x) + O(1)$.

Error term: Note that $t \longmapsto t e^{-c_1\sqrt{\log t}}$

is an increasing function

Hence $\int_{3/2}^{x} \frac{e^{-c_1\sqrt{\log t}}}{(\log t)^2}\, dt \leq e^{-c_1\sqrt{\log x}} \cdot x \underbrace{\int_{3/2}^{x} \frac{1}{t(\log t)^2}\, dt}_{\text{this is } O(1)}$

$\ll x \cdot e^{-c_1\sqrt{\log x}}$.

Our next goal is to study primes in arithmetic progressions ($a \bmod q$).

We already saw that if $(a,q)=1$, then there are infinitely many primes $\equiv a \pmod q$. More precisely, if $\chi \pmod q$ is non-principal, then $L(1,\chi)\neq 0$ (Dirichlet's theorem) and this implies distribution results of the form

$$\sum_{\substack{p\le x \\ p\equiv a(q)}} \frac{\log p}{p} = \frac{1}{\varphi(q)}\log x + O_q(1)$$

(This is Mertens restricted to APs, and of course Mertens is weaker than PNT)

We want to obtain stronger results, an equivalent of prime number theorem in arithmetic progressions; but also with uniformity in q!
(From now on, consider q a parameter alongside x).

Denote $\pi(x;q,a) = \sum_{\substack{p\le x \\ p\equiv a(q)}} 1$,

$$\theta(x;q,a) = \sum_{\substack{p\le x \\ p\equiv a(q)}} \log p, \quad \psi(x;q,a) = \sum_{\substack{n\le x \\ n\equiv a(q)}} \Lambda(n).$$

For a character $\chi \pmod q$, define

$$\psi(X;\chi) := \sum_{n \le X} \chi(n)\Lambda(n).$$

Then $\psi(X;q,a) = \dfrac{1}{\varphi(q)} \displaystyle\sum_{\chi} \overline{\chi}(a)\psi(X,\chi)$

(by orthogonality of characters)

Also, we see that for $Re(s) > 1$, we have

$$\sum_{n \ge 1} \frac{\chi(n)\Lambda(n)}{n^s} = -\frac{L'(s,\chi)}{L(s,\chi)}.$$

Hence, as in the proof of PNT we studied the properties of $\frac{\zeta'}{\zeta}(s)$, we now study $\frac{L'(s,\chi)}{L(s,\chi)}$,

in particular the location of zeros ρ of $L(s,\chi)$ play an important role.

<u>Lemma</u>: Let χ a primitive character modulo q (for $q \ge 2$). Then for $-1 \le \sigma \le 2$,

$$\frac{L'(\sigma+it)}{L(\sigma+it)} = \sum_{\rho : |s-\rho| \le 1} \frac{1}{s-\rho} + O(\log(q(|t|+2))).$$

(the sum is over non-trivial zeros ρ of $L(s,\chi)$ with multiplicity).

Proof: same as for $\frac{\zeta'}{\zeta}(s)$, but there is no pole at $s=1$, see exercises.

Next we need a variant of a zero-free region of $L(s,\chi)$:

Theorem: There exists an absolute constant $c>0$ such that if χ a character mod q, then the region $R_q = \left\{ s : \mathrm{Re}(s) > 1 - \frac{c}{\log(q(|\mathrm{Im}(s)|+1))} \right\}$ contains no zero of $L(s,\chi)$, unless χ is quadratic, in which case $L(s,\chi)$ has at most one, necesarilly real and simple zero in R_q.

Remark: Such a zero (if it exists), is called exceptional zero, or Siegel zero.

Proof: next time. As a corollary, we obtain:

Theorem: There exists a constant $C_4>0$ such that if $q \leq \exp(2C_4\sqrt{\log x})$ and $L(s,\chi)$ has no exceptional zero, then

$$\psi(x,\chi) = 1_{\chi=\chi_0}\, x + O(x\exp(-C_4\sqrt{\log x})).$$

If $L(s,\chi)$ has an exceptional zero β_1, then

$$\psi(x,\chi) = \frac{-x^{\beta_1}}{\beta_1} + O\left(x\exp\left(-c_4\sqrt{\log x}\right)\right).$$

<u>Proof</u>: Exercise.

<u>Remark</u>: This shows that PNT in AP would follow easily if we have no Siegel zeros. We need a way to control size of Siegel zeros.

Also note that PNT in AP follows for modulus q bounded by constant (since there are finitely many characters of modulus q, their real zeros are uniformly bounded away from 1). So we now can prove for example

$$\pi(x;10,1) = \frac{1}{4}\,\mathrm{li}(x) + O\left(x\exp\left(-c\sqrt{\log x}\right)\right).$$

There are ways to control the size of Siegel zeros (beyond the scope of this course!), and it is possible to prove PNT in AP uniformly for all $q \le (\log x)^A$, $\forall A > 0$:

Theorem (Siegel–Walfisz).
Let $A > 0$. There exists $c = c(A) > 0$ such that for all $q \le (\log x)^A$ and $(a,q) = 1$,

$$\psi(x;q,a) = \frac{x}{\varphi(q)} + O(x e^{-c\sqrt{\log x}})$$

and $$\pi(x;q,a) = \frac{\mathrm{Li}(x)}{\varphi(q)} + O(x e^{-c\sqrt{\log x}}).$$

It is possible to obtain stronger results if we are interested in average of error terms, rather than bounding each individual one.

Let $E(x,q) = \max_{(a,q)=1} \left| \psi(x;q,a) - \frac{x}{\varphi(q)} \right|$.

Theorem (Bombieri–Vinogradov)
Let x, Q be such that $\frac{x^{1/2}}{(\log x)^A} \le Q \le x^{1/2}$, for some $A > 0$.
Then $$\sum_{q \le Q} \max_{y \le x} E(y,q) = O(Q \cdot x^{1/2} (\log x)^5).$$

This shows that average error term is $O(x^{1/2} (\log x)^5)$, as good as RH.